The eigenvalue approach to groundwater modelling for resource evaluation at regional scale

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Abstract
The dynamic response of piezometric head at any location in an aquifer to time variation of regional land surface recharge and abstraction can be expressed as a linear system comprising only a few conceptual water storages. Model structure and parameters are related to aquifer characteristics and spatial pattern of recharge by the eigenvalues and eigenfunctions of a general analytical solution to the linearised Boussinesq equation. The model is implemented as a stochastic ARMA difference equation, independently for each location. This modelling approach is demonstrated for an aquifer of 2000 km² area, yielding additional information about unobservable recharge and aquifer boundaries.

INTRODUCTION

Evaluation of an aquifer as a water resource is concerned primarily with the effect on stored groundwater of pumped abstraction in relation to climatically driven, highly variable recharge processes. The resulting dynamic variations in stored groundwater determine environmental effects such as the low-flow regime of streams. The available data for such evaluations often comprise estimates of the history of land surface recharge from water balance models, and piezometric records at a few locations in a poorly understood aquifer. In these circumstances, finite difference or finite element aquifer models are at a disadvantage because of their demands on knowledge of aquifer properties and specification of boundary conditions. These models also have high computational requirements because the piezometric head is calculated for every element and the time step is constrained by numerical accuracy and stability.

The purpose of our modelling approach was to identify the dynamic behaviour of an aquifer as a storage reservoir, and to quantify the relative recharge contributions from rivers and through the land surface. These requirements favour solution of the groundwater flow equation in terms of the eigenvalues, which determine the dynamic response in continuous time, and the corresponding eigenfunction components of spatial variation. When the model is to be applied to only a few locations and for large time increments, the complete set of eigenvalues and eigenfunctions is not required (Sahuquillo 1983). Sloan (2000) uses a similar approach to demonstrate that only a few eigenvalues are required for adequate simulation of the dynamic behaviour of an aquifer considered as a lumped system.

MODEL THEORY

The following theory can be applied to the general two-dimensional aquifer problem (Sahuquillo 1983), but we illustrate the argument by means of a one-dimensional aquifer
(Fig.1) with fixed-head boundaries at different levels. The resulting piezometric surface, in the absence of land surface recharge, is defined by $\eta(x)$ and may also include the effect of existing pumped abstractions that are relatively steady.

Land surface recharge is vertical drainage from the vadose zone overlying the aquifer, which can be defined by a spatial distribution $f(x)$ multiplied by a time-varying magnitude $R(t)$. Pumped abstractions that are unsteady and not correlated in time with $R(t)$ contribute to model error. The response $u(x,t)$ to land surface recharge is the observed piezometric level $h(x,t)$ relative to the unobservable level $\eta(x)$:

$$u(x,t) = h(x,t) - \eta(x)$$  \hspace{1cm} (1)

Variations of $u(x,t)$ with time are assumed to be small in comparison to the aquifer depth, and therefore transmissivity $T(x)$ is only spatially variable, as is storativity $S(x)$. The governing equation for the aquifer shown in Fig.1 is:

$$\frac{\partial}{\partial x} \left[ T(x) \frac{\partial u}{\partial x} \right] + f(x)R(t) = S(x) \frac{\partial u}{\partial t} \hspace{1cm} x \in (0,L), \hspace{0.5cm} t \in (0,\infty)$$  \hspace{1cm} (2)

with boundary conditions:

$$u(0,t) = 0, \hspace{0.5cm} u(L,t) = 0 \hspace{0.5cm} t \in (0,\infty)$$  \hspace{1cm} (3)

Sloan (2000; Appendix A) shows that a general solution of (2) can be expressed in terms of the eigenvalues $\lambda_i$ and eigenfunctions $p_i(x)$ derived from the physical characteristics of the aquifer as:

$$u(x,t) = \sum_{i=1}^{\infty} p_i(x)w_i(t)$$

$$\frac{dw_i(t)}{dt} + \lambda_i w_i(t) = c_i R(t)$$

$$w_i(0) = W_i$$  \hspace{1cm} (4)

The general solution (4) can be illustrated in conceptual form (Fig.2), for a location $x = X$, as the weighted output $u(X,t)$ from an infinite set of linear reservoirs in parallel, with input $c_i R(t)$ to the $i$th reservoir. The water content of each conceptual reservoir is represented by $w_i(t)$, with initial value $W_i$. The mean residence times are given by $\frac{1}{\lambda_i}$, the reciprocals of the eigenvalues. Sloan (2000) reports that only a few of these reservoirs, corresponding to the smallest eigenvalues, are required to simulate adequately the dynamic behaviour of an aquifer. We use an additional reservoir (Fig. 2) in series with the parallel set to simulate storage in the vadose zone and the dynamic effect of leakage through aquitards.

We implemented the model structure of Fig. 2 within Microsoft Excel by converting (4) to a stochastic difference equation for each observation well, which relates monthly totals of recharge $R_k$, estimated from a water balance model, to monthly observations of groundwater level $h_k$. Only the first three eigenvalues were considered, so that with the inclusion of the vadose zone element the resulting difference equation was of fourth order. The model structure of Fig. 2 can be expressed in discrete time intervals by means of z-transforms as:
\[ u_k = \left[ \frac{\beta_1}{(1 - \alpha_1 z^{-1})} + \frac{\beta_2}{(1 - \alpha_2 z^{-1})} + \frac{\beta_3}{(1 - \alpha_3 z^{-1})} \right] \left( \frac{1 - \alpha_4}{(1 - \alpha_4 z^{-1})} \right) R_k \]

\[ h_k = u_k + \eta + e_k \]

in which \( u_k + \eta \) is the model estimate of the observation \( h_k \), and \( e_k \) is an error term. The values of \( \alpha \) are determined from the eigenvalues \( \lambda_i \) \((i=1,2,3)\) and the mean hydraulic residence time \( \tau \) of the vadose zone, for the data observation interval \( \Delta t \), as:

\[ \alpha_i = \exp(-\lambda_i \Delta t) \quad i = 1,2,3 \]

\[ \alpha_4 = \exp\left(\frac{-\Delta t}{\tau}\right) \quad (6) \]

The parameter \( \beta_i \) is a function of \( c_i, p_i(x) \) and \( \lambda_i \) in (4). For computation, (5) is converted to an autoregressive-moving-average (ARMA) difference equation:

\[ u_k = a_1 u_{k-1} + a_2 u_{k-2} + a_3 u_{k-3} + a_4 u_{k-4} + b_1 R_k + b_2 R_{k-1} + b_3 R_{k-2} \]

\[ e_k = h_k - u_k - \eta \quad (7) \]

for which the relationships between coefficients \( a_i, b_i \) in (7) and \( \alpha_i, \beta_i \) in (5) are obtained by multiplying out (5) and equating powers of \( z^{-1} \). The error term series \( e_k \) is the basis for model calibration and analysis of performance. The steady-state gain (ssg) of (7) is a useful measure of the mean piezometric response (m) at the particular location to mean land surface recharge (mm mth\(^{-1}\)), and is calculated from:

\[ ssg = \frac{b_1 + b_2 + b_3}{1 - a_1 - a_2 - a_3 - a_4} \quad (8) \]

In general, the \( ssg \) increases with distance from fixed-head boundaries, and is a useful parameter for mapping and analysing aquifer characteristics.

Model calibration is conducted on the parameters \( \lambda_i, \beta_i, \tau \) and \( \eta \) in (5) and (6) because this form has minimal parameter interdependence and physically realistic initial values can be set. An important contribution to groundwater modelling is that the eigenvalues are the same at every location in the aquifer, and therefore calibrated values of \( \lambda_i \) from locations with good data records may be transferred to assist model calibration at locations with sparse data. We used the optimisation routine “solver”, provided within Microsoft Excel, for model calibration at each location for which a groundwater level record was available.

**MODEL DEMONSTRATION**

We demonstrate the eigenvalue approach with some results obtained from a water resource study of the aquifer underlying part of the Canterbury Plains in New Zealand. The plains are 160 km long and 50 km wide between the mountains of the Southern Alps and the Pacific Ocean, formed by coalescing glacial outwash and alluvial deposits up to 600 m thick (Brown 2001). The study area is the 2000 km\(^2\) Central Plains region bounded by the Waimakariri and
Rakaia Rivers that traverse the plains from mountains to sea. These large braided rivers are perched on alluvial fans and their interaction with the underlying aquifer is not completely understood, so that it is difficult to define boundary conditions for the aquifer.

The recharge series $R_k$ for the region, as monthly totals, was estimated from a daily water balance model. This series, with mean annual total of 204 mm, was applied at all locations even though there is a rainfall gradient from about 950 mm y$^{-1}$ near the mountains to about 650 mm y$^{-1}$ near the ocean. The effect of this spatial variation in rainfall on land surface recharge is assumed to be time-invariant, as illustrated by $f(x)$ in (2), and therefore $R_k$ is simply scaled by the model coefficients at each location.

The model (5)-(7) was calibrated independently at 14 observation wells in the region, with up to 28 years record of groundwater level at monthly intervals. The first four observations of groundwater level were used to compute the initial values of $u_{k-i}$ in (7). The objective function in all cases was to minimise the sum of $e_k^2$.

RESULTS

We have selected four observation wells with the longer and more reliable records to present the model results in Table 1. The well locations form an approximate rectangle of 20 km x 15 km. Some of the remaining records were too short to capture the dynamic response, or were strongly influenced by local abstraction for spray irrigation and by recharge from surface irrigation. In the course of our resource study the eigenvalues were transferred from other locations, as supported by the theory, to assist with modelling these additional effects, but these results are not discussed in this paper.

DISCUSSION

The results in Table 1 are presented in order of the piezometric response $ssg$ of the model at each well, which is theoretically related to the distance of the well from a fixed-head boundary. Well L35/0163 is adjacent to the Waimakiriri River on the northern boundary of the study region, but the high value of $ssg$ suggests that the river is not a fixed-head boundary at this location. In contrast, Well M35/1080 is near the lower reaches of this same river and the relatively low value of $ssg$ suggests a more direct connection between the aquifer and the river at this location. The intermediate values of the other two wells are appropriate for their locations relative to the river and ocean boundaries.

The dominant residence time $\lambda_1^{-1}$ at all wells is not significantly different from a value of 20 mth. This means that the aquifer can be considered, for management purposes, as a reservoir with a mean residence time of 20 mth, which receives a time-varying input of about 200 mm y$^{-1}$ of recharge over the 2000 km$^2$ region. This concept is useful for quantifying the effect on lowland stream regimes of future abstraction from the aquifer. Only one other eigenvalue was significant, and then only for the two wells nearest to the fixed-head boundaries as indicated by the smaller values of $ssg$. The values of residence time $\lambda_2^{-1}$ for Wells M36/0255 and M35/1080 are not significantly different from 0.6 mth.

Comparison of the relative magnitudes of $\beta_1$ and $\beta_2$ for Wells M36/0255 and M35/1080 demonstrates the increasing importance of the larger eigenvalues (smaller residence times) in
explaining aquifer dynamics near fixed-head boundaries, as indicated by the smaller values of \( ssg \). The physical explanation is that initial response to perturbations of the piezometric surface is rapid near these boundaries because of the shorter drainage path, but subsequently is determined by water draining from the interior of the aquifer.

The residence time \( \tau \) of the vadose zone element varies with location from about 1 to 5 mth, and is loosely associated with depth to groundwater (data not shown). We know that an aquitard overlies the aquifer at Well L36/0092, and therefore slow drainage of perched groundwater may be a significant component of the 5-month residence time. It is worth noting that \( \tau \) is a measure of the hydraulic response of the vadose zone, which is much less than the transit time of a tracer particle.

The values of \( \eta \), relative to mean sea level (amsl), contribute to defining the piezometric surface due to river recharge and steady abstractions. We propose using this information, in conjunction with lumped aquifer properties obtained from the dynamic analysis of the surface recharge response, to estimate the approximate magnitudes and locations of river recharge to the aquifer.

The explained variance \( R^2 \) provides a measure of the ability of the eigenvalue modelling approach to simulate the dynamic behaviour of an aquifer. However, there is still considerable information contained in the error series \( e_k \) at each of the observation sites, and further analysis may assist in quantifying the effects of abstractions and recharges not previously considered.

**CONCLUSIONS**

The benefits of the eigenvalue approach to groundwater modelling depend on aquifers behaving as distributed dynamic systems with a high degree of interdependence among the components. This means that the dynamic effects of heterogeneous aquifer properties and spatial variations in recharge can be condensed into a much smaller number of independent model parameters. The most significant of the dynamic system parameters are observable at any location in the aquifer, and useful information can be obtained from simple models calibrated with time-series of piezometric head at single locations. The reliability of this information is quantifiable in terms of the error component of a stochastic difference equation. Some of the model parameters are location-specific, and these can provide information about unobservable sources of recharge and the characteristics of aquifer boundaries.

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REFERENCES

Table 1 Parameters of the linear system model of aquifer response to land surface recharge.

<table>
<thead>
<tr>
<th>Observation well reference</th>
<th>L35/0163</th>
<th>L36/0092</th>
<th>M36/0255</th>
<th>M35/1080</th>
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<tr>
<td>Piezometric response ssg (m mm⁻¹ mth)</td>
<td>1.22</td>
<td>1.12</td>
<td>0.30</td>
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<td>Dominant residence time $\lambda_1^{-1}$ (mth)</td>
<td>18.9</td>
<td>20.4</td>
<td>19.2</td>
<td>20.4</td>
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<td>Secondary residence time $\lambda_2^{-1}$ (mth)</td>
<td>-</td>
<td>-</td>
<td>0.61</td>
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<td>Dominant coefficient $\beta_1$</td>
<td>0.063</td>
<td>0.053</td>
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<td>Secondary coefficient $\beta_2$</td>
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<td>-</td>
<td>0.014</td>
<td>0.034</td>
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<td>Vadose zone residence time $\tau$ (mth)</td>
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<td>4.8</td>
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<td>Steady recharge datum $\eta$ (m amsl)</td>
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<td>Model performance $R^2$</td>
<td>0.873</td>
<td>0.911</td>
<td>0.852</td>
<td>0.823</td>
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**Fig. 1** One-dimensional heterogeneous aquifer with time-varying land surface recharge and fixed-head boundaries.

**Fig. 2** Linear-system water storage model of aquifer response to land surface recharge.
Fig. 1 One-dimensional heterogeneous aquifer with time-varying land surface recharge and fixed-head boundaries.
Fig. 2 Linear-system water storage model of aquifer response to land surface recharge.