A SIMPLE APPROACH TO VALUING THE DELIVERY OPTIONS IMPLICIT IN THE US TREASURY BOND FUTURES CONTRACT

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November 1998

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ISSN 1174-5045
ISBN 1-877176-36-2
I would like to thank Mike Tomas and Dan Grombacher from the Chicago Board of Trade (CBOT) for their helpful comments. I would also like to acknowledge financial assistance from the Educational Research Foundation of the CBOT.
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1.0 Introduction

The T-Bond futures contract has traded on the Chicago Board of Trade (CBOT) for approximately 20 years, and is now the most actively traded derivative product in the world. A voluminous amount of research has accordingly attempted to analyse the contract, and many have tried to establish a theoretical price which fully reflects all of the contract specifications. One factor that complicates this process is the series of delivery options which are embedded in the T-bond futures contract, and which give the short position some flexibility in their choice of the particular bond to be delivered as well as when delivery should be made. While the valuation of both the futures (and futures options) should obviously consider the value of these implicit delivery options, there is no existing methodology with which to estimate the values on an aggregate, and consistent basis.

All of the previous methods for valuing the delivery options can be conveniently separated into two main groups, each having some distinct advantages and disadvantages. The first uses historical price data to simply measure the payoffs that would have accrued to an investor who follows a trading strategy that optimally utilises the delivery options available to a short futures position. Such an approach can usefully quantify the ex-post value of the options at expiration, and has the advantage that it does not require use of a particular option valuation formula. It is difficult however to generalise these results and does not provide any way to value the options prior to expiration on an ex-ante basis. That is the aim of the second approach, which specifies and applies a pricing model in an attempt to measure the amount by which futures prices should be bid down in equilibrium to reflect the options’ value. The success of the second approach obviously depends on the suitability of the specified valuation formula.

It is clear that the current value of the delivery options is some function of the possible future changes to the values of the T-bonds that underlie the futures contract. These values are themselves dependant on innovations in the term structure of interest rates between the valuation date and the last day of trading in the futures contract. Accurate estimates of the value of the embedded options are therefore clearly reliant on the adequacy of the term structure model used in the analysis.
The most general of these models is that of Heath, Jarrow and Morton (1992) (HJM). Unlike those that use spot interest rates or bond prices, the HJM approach is based on a stochastic process for the entire forward rate curve, and implies a spot rate process which is generally non-Markov. Using a no-arbitrage argument, HJM show that all future innovations in forward rates (in a risk-neutral environment) can be expressed as a function of just the instantaneous forward rate volatilities. Analogous to the Black-Scholes valuation model, the prices of all claims contingent on the future evolution of the term structure are therefore also completely specified by the description of a volatility function. By using the initial term structure as an input to the model, it has the desirable feature of automatically ensuring that model-based values closely match those observed in the market.

Unfortunately, most of the HJM models are path dependent. As there are generally no closed-form solutions for contingent claim prices, implementation of the valuation procedure must be based on one of a number of numerical methods. Discrete-time approximations of American-style claim values are commonly handled using trees - a binomial tree for a one-factor model, a trinomial tree for a two-factor model, and so on. These trees are usually non-recombining, meaning that the total number of terminal nodes in the tree grows exponentially with the number of time-steps. Unfortunately, this means that the most general HJM models are unsuitable for valuing the delivery options because they require a prohibitive amount of computational effort.

In order to make the analysis tractable it is necessary to choose a forward rate volatility structure which implies a Markov process for the spot rate, and which therefore gives a tree which is recombining. The simplest of all cases is to assume that the volatility function is a constant, meaning that the HJM model reduces to the continuous-time limit of the Ho and Lee (1986) model. Discrete-time versions of that model can then be used to generate values for the delivery options under a range of forward rate, and forward rate volatility scenarios.

The remainder of this study proceeds as follows. Section 2 provides a description of the T-Bond futures contract, with a particular emphasis on the nature of the embedded delivery options. The Ho-Lee model is briefly reviewed in Section 3, using the fact that that model is a special case of the more general HJM term structure model. Section 4 outlines an approach to
valuing the delivery options in the HL framework, which is then used to present some simulated option values in Section 5. A brief summary of the study is given in Section 6.

2.0 The Treasury Bond Futures Contract

The asset underlying the CBOT Treasury Bond Futures contract is a T-Bond with face value of $100,000 and a ‘notional’ coupon rate of 8%. As at the beginning of the contract delivery month, any such bond which has at least 15 years to maturity (or, if callable, has at least 15 years to the first call date) may be delivered to settle the contract. The contract standards are designed to allow a broad spectrum of deliverable bond issues which vary widely with respect to maturity and coupon interest rate. Consequently, there are usually at least 20 outstanding Treasury bonds that qualify. T-bond futures prices are quoted in dollars and thirty-seconds of par value of $100. Contract expiration months are March, June, September, and December and extend out for two years. Delivery can be made on any business day of the contract month, although trading in the futures contract ceases on the business day prior to the last seven days of the expiration month.

At delivery, the party with the short futures position receives a cash amount from the long position in exchange for one of the bonds in the deliverable set. The choice of which of the eligible bonds is delivered is at the discretion of the short position. In order to adjust for differences in the spot market values of deliverable grade bonds, the CBOT has established a conversion factor invoicing system which is intended to reconcile these differences by adjusting every bond to provide a yield of approximately 8%. Conversion factors are computed as the price that bonds with a unit face value, and a coupon rate and time to maturity equal to those of the deliverable bonds, would have if it were priced to yield 8% (compounded semi-annually).

The assumed bond maturity and the times to the coupon interest payment dates are rounded down to the nearest three months. Thus, if the adjusted maturity of the bond is an exact multiple of half year periods, then the next coupon payment is assumed to occur in six months from the calculation date. Otherwise, the first coupon payment is assumed to be paid after three months and accrued interest is then subtracted from the hypothetical bond value.
This procedure allows the CBOT to produce comprehensive tables of conversion factors that are fixed for all of the bonds in the deliverable set.

The T-bond futures contract is also the underlying instrument for a series of put and call options. Option contract prices are quoted in sixty-fourth’s of a point ($15.625 per contract), have strike prices which bracket the current T-bond futures price, and are listed in the front month of the current quarter plus the next three contracts of the regular quarterly cycle (March, June, September, and December). All options expire in the month prior to the stated contract month. They cease trading at noon on the last Friday proceeding (by at least five business days) the expiration month. For example, the last days of trading in the April 1996 and June 1996 futures options contracts are March 22 and May 17, respectively. All options are American style.

Determination of a precise price for the T-bond futures contract (and therefore the futures options contract) is complicated by a number of delivery options afforded the short position. If these options have value, then we should expect that market equilibrium requires that the futures price be bid down by the value of those options. The difference between the theoretical value of the futures contract, exclusive of the option features, and the quoted futures price constitutes an implicit payment by the short position.

If the short position decides to settle his position by delivery, the cash amount received depends on both the current quoted futures price and which of the eligible bonds is delivered. Specifically, the invoice amount is determined as; Invoice Amount = (Quoted Futures Price * Conversion Factor) + Accrued Interest. The short position purchases the bonds for delivery in the cash market at a cost of; Bond Cash Price = Quoted Bond Price + Accrued Interest. Rationally, the short position will attempt to maximise the difference between the cash inflows and outflows, and thus will choose to deliver the bond for which the following equality is greatest (and for which the associated bond is called the cheapest-to-deliver bond): Short Cash-Flow = (Quoted Futures price * Conversion Factor) - Quoted Bond Price.

This flexibility constitutes what is known as the quality option, and is valuable because the system used to adjust the invoice prices for bonds with different maturities and coupons (via the conversion factors) is imperfect. If current bond yields exceed 8%, the conversion factor...
system tends to favour delivery of relatively low coupon, long maturity issues. Conversely, bonds with high coupons and short maturities are liable to be cheapest-to-deliver when yields are less than 8%, while the delivery of long term (short term) bonds is favoured when the yield curve has positive (negative) slope. Some cash market biases may also have an impact on which of the deliverable set is cheapest-to-deliver. There appears to be a general preference in the cash market for low coupon bonds, or bonds for which it is possible to separate the coupon and principal payments (stripped bonds). As such features often attract a premium, it is unlikely the corresponding securities will be identified as the cheapest-to-deliver issue.

The mechanics of the delivery system give rise to three other implicit options for the short party, all of which relate to the timing of delivery. First, delivery can be made on any business day of the contract month, and the value of this flexibility is commonly described as follows. Suppose that the cheapest-to-deliver bond has a coupon rate that exceeds the rate that could be earned on the cash payment received from immediate delivery. Then the short should keep the position open for as long as possible because accrued interest is greater than the opportunity cost. If, on the other hand, the current market interest rate exceeds the daily adjustment to the invoice price then the short would be best suited by an early delivery. Ignoring the impact of the other embedded options, this timing (or accrued interest) option should encourage all deliveries to occur either early or late in the delivery period.

The wildcard option arises because of the differences in the closing times of the spot and futures markets. Trading in the futures market ceases at 2 p.m. (CST) while trading in the Treasury bonds themselves continues for several hours. On each day of the delivery period, the short position has until 8 p.m. (CST) to notify the Clearing Corporation of an intention to deliver. This creates a daily six hour window during which the short may potentially take advantage of changes in cash market prices. If the spot price declines after the close of the futures markets (at which time the invoice price is fixed), the short may decide to purchase the underlying security and issue a notice of intention to deliver. However, optimal delivery is not guaranteed by a price decline in the spot market unless the price drop is significant enough to dominate the cumulative value of all subsequent wildcard options that are forfeited by immediate delivery.
The final delivery option of the short position is referred to as the *end-of-month option*. It arises because trading in the futures contract ceases on the eighth to last business day of the expiration month, but deliveries may continue on each of the remaining business days. As the futures settlement price is fixed over that period, the short position may benefit for two reasons. First, the short has some discretion as to when the bond is delivered and, similar to the wildcard option, can profit if the spot bond prices decrease after the futures contract has ceased to trade. Unlike the wildcard option however, the short position has an obligation to settle eventually and the expected profits from waiting to settle are therefore symmetric. That is, absent any special knowledge, cash bond prices are just as likely to increase during the end-of-month period. Expected profits from waiting should therefore (on average) be zero and the option to delay delivery should have no value in this context.

The end-of-month option may still have value however, if the delay in delivery allows the short to change which bond is delivered in the last seven days of the delivery month. On the last day of trading in the futures contract, we can assume that the settlement price is related to the cheapest-to-deliver bond on that day. If the cheapest-to-deliver bond changes over the subsequent seven day period, then the short position can profit by substituting that bond in the delivery process. If this option has any value, then we should expect that the settlement price on the last day of futures trading will be less than the value of the cheapest-to-deliver bond at that time.

### 3.0 The Ho-Lee Model

Ho and Lee (HL) were the first to derive a model for interest rate derivatives under the assumption that the initial term structure is exogenous. They derived a process for feasible subsequent term structure movements that precludes any arbitrage opportunities within the set of currently traded securities. This contrasts with the traditional equilibrium models, where the initial term structure is determined endogenously and with no guarantee that the model ‘output’ will exactly match observed market data. All equilibrium models also suffer from their explicit reliance on unobservable quantities such as the ‘market price of risk’, or equivalently, the expected rates of returns on discount bonds. As mentioned above, the HL
model is a special case of the one-factor Heath, Jarrow and Morton (1992) model. The following treatment of the HL model is discussed in this light for ease of exposition.

In continuous-time, HJM specify the evolution of forward rates in a one-factor, pseudo (or risk-neutral) economy according to:

$$df = \mu_f(t,T)\,dt + \sigma_f(t,T)dW(t),$$

where $\mu_f(t,T)$ and $\sigma_f(t,T)$ are the drift and volatility parameters that could depend on the term structure itself, $dW(t)$ is a scalar standard Wiener process used to model the single source of uncertainty, and $f(t,T)$ is the instantaneous forward rate observed at time $t$, for the time increment beginning at date $T$. HJM show that under a risk-neutral probability measure, the choice of the volatility function completely determines all claim prices because each choice uniquely determines the drift coefficients. That is, by assuming a no-arbitrage economy, one is able to generate a tree describing the possible future paths for the forward rate without reference to the drift term, $\mu_f(t,T)$, in the specified forward rate process.

The procedure only requires knowledge of the initial term structure, and an explicit modelling of the volatility function of the forward rate curve. As such, HJM suggest that their model is analogous to the Black-Scholes formula for pricing equity options, where the required inputs are the current asset value and the volatility of the asset value, but not quantities such as the market price of risk or the expected rate of return on the asset. Unlike the Black-Scholes model however, the HJM procedure requires that the volatility function must describe the stochastic evolution of the entire term structure curve. Once a specific form for the function is chosen, volatilities can be either estimated statistically from historical term structure movements, or implied from a set of observed market prices for some interest rate claims.

Note that the functional form of the volatility process is not restricted to some particular specification. In fact, in the most general case, the function $\sigma_f(t,T)$ can be dependent not only on time, but also on both the past and current levels of the term structure. This generality does come at a cost: it means that forward rates of all maturities cannot be represented as functions of a small number of state variables that have evolutions governed
by Markovian processes. Thus, in practical terms, it is not possible to represent the future paths for the

forward rate curve using a recombining tree. After \( n \) time-steps in a one-factor model, the HJM tree contains \( 2^n \) terminal nodes compared to just the \( n + 1 \) nodes required to describe a Markov process using a standard binomial tree. As ‘bushy’ trees grow exponentially fast, the chosen number of time-steps must be quite small to ensure that the computation time does not become prohibitive.

The problem of path dependency can be overcome by choosing one of a number of simple volatility functions. The simplest of all is the case in which \( \sigma_{f}(t,T) \) is assumed to be a constant, \( \sigma_{f}(t,T) = \sigma \). This gives rise to the Ho and Lee model\(^1\), which describes the forward rate evolution as a Markov process and which can therefore be represented in a recombining tree. Of course, the computational efficiency gained via this restriction comes at a cost of reintroducing the deficiencies of the Ho and Lee approach. These include the possibility of negative interest rates and the inability to represent non-parallel shifts in the forward curve.

Efficient HJM trees can be constructed using a parameterisation of the discrete model given in HJM (1991). For the one-factor model, assume that the forward rate process is described by:

\[
f(t + \Delta,T) = \begin{cases} f(t,T) - \sigma(t,T,f(t,T))\sqrt{\Delta} + \delta(t,T)\Delta & \text{with prob } q_i, \\ f(t,T) + \sigma(t,T,f(t,T))\sqrt{\Delta} + \delta(t,T)\Delta & \text{with prob } 1-q_i, \end{cases}
\]

where \( f(t,T) \) is the continuous forward rate maturing at time \( T \) as observed at time \( t \), \( \sigma(\cdot) \) is the forward rate volatility function, \( \Delta \) is the length of the time-step, and \( \delta(t,T) \) is an adjustment term chosen to ensure that the forward rate process is arbitrage free under the arbitrarily chosen pseudo probabilities. That is, \( \delta(t,T) \) is chosen to meet the following no-arbitrage condition:

\[
\bar{E}_{t} (P(t,t + \Delta) P(t + \Delta,T)) = P(t,T)
\]

\(^1\) HJM (1992) also present the specific volatility functions which give the special cases of the Vasicek (1977), Cox, Ingersoll, and Ross (1985), and Hull and White (1992) models.
where, \( P(t,T) \) is the time \( t \) price of a discount bond that matures at time \( T \), and \( \tilde{E} \) denotes expectations under the pseudo probabilities. HJM (1991) show that the solution for the adjustment term is:

\[
\delta(t,T)\Delta = \frac{\partial}{\partial T} \ln \left( \cosh \left( \int_{t}^{T} \sigma(t,u,f(t,u)) \sqrt{\Delta} \, du \right) \right).
\]

(4)

This conveniently reduces to a very simple term when the volatility function is assumed to be a constant.

Computing prices for contingent claims is very straightforward in the HJM framework. Based on realisations of HJM forward rate trees from (2), it can be shown that the no-arbitrage futures price at time \( t \) for a contract that matures at time \( T \) is:

\[
F_t(t) = \tilde{E}[F_t(T)]
\]

(5)

That is, futures prices are martingales under the risk-neutral expectations operator, \( \tilde{E} \).

HJM have also shown that the time \( t \) value of a European option which has a terminal payoff of \( C_T \) is a martingale relative to what they term a money market account. This implies that:

\[
C_t = \tilde{E}[C_T / B(T)] / B(t)
\]

(6)

where \( B(t) \) is defined to represent the time \( t \) value of a dollar invested in a money market account at time zero, and which grows at the spot rate, i.e.,

\[
B(t) = \exp \left( \int_{0}^{t} r(u) \, du \right)
\]

(7)

The current value of the European option is then simply equal to the expected terminal payoff, discounted back to today at the (average) riskless rate. This is very similar to the well-known valuation approach of Cox, Ross, and Rubinstein (1979), but here the discounting rate is obviously not a constant. Instead, the spot rate at each node in the tree is dependent on the unique path taken by the forward curve leading to the node.
4.0 Incorporating Delivery Options in a Lattice Framework

Efficient valuation of American style options generally requires the use of either a tree or lattice based approach. The potential future paths of the underlying asset’s value are generated (extending out to the expiration date of the option), and the option is valued at each node in a recursive fashion. For American options, a tree is required so that it is possible to incorporate the value of possible early exercise throughout the option’s life. In the case of futures options, the underlying asset (futures contract) is itself a derivative security and valuation of the option requires the generation of a tree of futures prices extending out to expiration of the futures contract. Such a procedure is reasonably straightforward when the futures contract is simple, and does not contain any delivery options like those embedded in the T-bond contract. Unfortunately, incorporating the value of the delivery options into the tree of futures prices poses numerous theoretical and practical problems.

Some of these issues are discussed by Fleming and Whaley (1994), who examine wildcard option valuation in a binomial lattice framework. Specifically, they show how to include wildcard options into the valuation of the S&P 100 index option contract (OEX) traded at the Chicago Board Options Exchange, such that the option price at any node on the tree reflects the value of all subsequent daily wildcards. The wildcard features embedded in this contract are similar in nature to those attached to the T-bond futures contract. Here, the flexibility arises because the settlement price of the option is set equal to the S&P 100 index level at 3.00 p.m. (CST), which is the close of trading at the NYSE. Option positions may however wait until 3.15 p.m. before deciding whether or not to exercise their options. For call options, a large market decline during the wildcard period may result in an optimal exercise if the losses avoided by doing so exceed any forfeited premium.

In theory, the value of the wildcard opportunities can easily be included in a binomial tree of call or put option prices. It simply requires that the length of each time-step be set to match the length of the wildcard period, and that the usual early exercise bounds be modified at wildcard nodes to account for the pre-existing settlement prices. Practically implementing this adjustment is of course problematic because the number of nodes required in the tree quickly grows to unreasonable levels. For instance, in the case of the S&P 100 index options, the wildcard period spans a 15 minute interval and the above technique therefore requires 96
time-steps each day (24 hours and 4 intervals per hour), and 2,880 time-steps over the life of just a thirty-day option. The associated computing requirements are obviously prohibitive, and lead Fleming and Whaley (1994) to suggest a simplified, and therefore feasible alternative. This is reviewed at some length here because a similar procedure may be useful for dealing with the wildcard option in the T-bond futures contract.

The key to their procedure is to set the length of the time-step such that the nodes in the binomial tree coincide with the end of each wildcard period. Because the OEX contract contains a wildcard feature at the end of all trading days, this adjustment requires that the total number of nodes be set equal to the number of days to the expiration of the option. The resulting number of total nodes in the tree is far more manageable, even for the longest-dated traded contract. To incorporate the wildcard feature, the standard ( wildcard exclusive) early exercise bounds for an American call option are adjusted as follows. At each node in the tree, call value is determined as:

$$\begin{align*}
C_n^j &= \max\{S_n^j - X, \ E(C_n^j) + W_n^j\},
\end{align*}$$

where, $C_n^j$ is the value at node $j$ and time-step $n$, $S_n^j$ is the corresponding stock price, $X$ is the exercise price, $E(C_n^j)$ is the expected call value at that node (based on the discounted call values from either an up or down movement in the next period), and $W_n^j$ is the wildcard option value at the node in question. Inclusion of the wildcard value now means that the call option holder’s early exercise decision rule is slightly altered. Here, at the end of day $n$, the proceeds from immediate exercise ($S_n^j - X$) are compared with the value of an unexercised option ($E(C_n^j)$) plus the value of the wildcard option on that day. Because the formula in (8) is used recursively, $E(C_n^j)$ implicitly incorporates the value of all future wildcard options that remain between day $n$ and the expiration day.

Implementation of (8) still requires some method for valuing the daily wildcard, $W_n^j$. Normally, the payoff structure from any option is considered at the beginning of the relevant period, which in this case would be 15 minutes prior to the end of each day and is denoted as time $n - w$. At that time, the payoffs to the wildcard option can be written as:
where $S_{n-w}$ is the stock price at the start of the wildcard period (and is therefore known), and $\tilde{C}_n$ is the uncertain value of the call option at the end of the 15 minute period, which implicitly depends on the uncertain ending stock price. If stock prices decrease dramatically during the wildcard period, the ending value of the call option is likely to decrease to such an extent that early exercise is likely as we approach time $n$, $(S_{n-w} - X > \tilde{C}_n)$. Conversely, the wildcard option will not be used if the stock price increases because the ending call value will reflect the higher underlying value and exceed the proceeds from early exercise $(S_{n-w} - X < \tilde{C}_n)$.

However, because the binomial time-steps are chosen to coincide with the end of the wildcard period, some adjustment to these payoffs are required. As at time $n$, the ending call value $(C_n)$ is known but the earlier settlement price $(\tilde{S}_{n-w})$ is not, requiring that the wildcard payoff be rewritten as:

$$W_n^w = \max[(S_{n-w} - X) - \tilde{C}_n, 0],$$

(9)

The value of the wildcard option at time $n$ can then be estimated by approximating the asset price $(\tilde{S}_{n-w})$ distribution at time $n-w$. Although Fleming and Whaley (1994) suggest that this may be done using a second binomial tree (moving backward from each node at each time-step), they choose to use the lognormal asset price distribution assumption and derive a closed-form solution for the wildcard option value that is similar to the Black-Scholes formula. Working backwards from expiration of the OEX option, this solution is used at each node to add the daily wildcard value to that of the call or put option, which in turn is computed by discounting the expected future value. In this way, option values at all nodes incorporate both the value of the wildcard at that particular node and the cumulative value of all subsequent wildcards.

At first glance it might appear that the recursive nature of the HJM model should allow incorporation of the various delivery options embedded in the T-Bond futures contract in a similar way to that used by Fleming and Whaley. Working back through the tree, one can add or implicitly account for the delivery options as they occur at each time-step. As the futures
prices at each node are dependent on some relevant futures prices at the next time-step, the recursive procedure ensures that the computed price at any given node accounts for the value of all subsequent options. Specifically, one should first estimate the values of the end-of-month options and then combine these with the previously determined no-arbitrage futures prices at a time-step that coincides with the end of trading in the futures contract. The option-exclusive futures value at each terminal node is determined as the value of the cheapest-to-deliver bond, which is itself determined at each node using the generated vector of forward interest rates. Proceeding backwards from this time-step gives futures values which fully reflect the value of the end-of-month option. If the length of time-step is also chosen such that earlier nodes coincide with the end of each daily option during the delivery period, then those values can be established and included in the tree of futures prices.

Practical implementation of this entire procedure is however impossible, and some major simplifications are required to deal with even one or two of the delivery options. To see this, consider attempting to generate an HJM tree which starts at the beginning of the delivery period, and which is structured such that each time-step corresponds with the end of a delivery day. If we assume 30 days from the start of the delivery month to the last possible delivery day, the tree would require 30 time-steps and the corresponding number of terminal nodes will be over one billion for just a one-factor model. The computational requirements for such models are obviously restrictive, and are exacerbated if the time horizon is extended to include trading months prior to the delivery month.

That is why this study imposes a deterministic volatility function on the (one-factor) forward rate process to get a continuous-time limit of the Ho and Lee (1986) model. The resulting Markov interest rate process yields a discrete-time approximation that is path-independent, and gives a corresponding binomial tree that contains a manageable number of nodes. The unfortunate drawback of this simplifying restriction is that it eliminates the rich structure of potential movements in future interest rates that are possible under a more general volatility process in a two-factor model. Nonetheless, even if all of the delivery options can only be incorporated into the Ho-Lee model, the resulting examination of their cumulative impact on equilibrium futures prices is still useful. The analysis may indicate that the value of some of the embedded options is insignificant, and perhaps a practical model for futures prices can disregard the associated complications.
With these computational issues in mind, it is now appropriate to discuss some preliminary methods by which to incorporate the impact of the four delivery options into an HL tree of futures prices. Each option is discussed in the order that they should be included in the tree, starting with those that occur at the end of the contract period.

### 4.1 The End-of-Month Option

Recall that the futures contracts for delivery in a specific contract month cease to trade on the eighth business day prior to the end of that month. Contracts open after this date must be delivered on one of the last seven business days. Since the settlement price for the last trading day determines invoice prices for that day and the seven subsequent business days, the short position can observe spot bond prices over this interval and choose when and what to deliver.

Gay and Manaster (1986) suggest that the value of this flexibility will likely arise from the ability of the short position to switch between bonds if the optimal delivery bond changes during the end-of-month period. For every delivery period, this value is observed on an ex-post basis as the difference between the futures settlement price on the last day of trading and the concurrent price of the cheapest-to-deliver bond.

One way to estimate the value of the end-of-month option on an ex-ante basis (and then incorporate that value into futures prices) is as follows. Assume that the HL tree is generated in the usual manner such that the terminal nodes correspond to the end of trading in the futures contract. At each of these nodes, the corresponding vector of forward rates can be used to compute the value of all deliverable bonds, and the no-arbitrage futures price initially set equal to the value of the cheapest-to-deliver bond. Now, to account for the potential benefits of the end-of-month period, each terminal node can be used as the starting point of a secondary tree which extends out to the last day of the delivery month. If the secondary trees contain the same information as the primary tree, then one can determine the cheapest-to-

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2 Thus, in most applications, the length of the time-step for the secondary trees will be considerably shorter than that used in the primary tree. Note also that the computing requirements for this procedure are the same as those for a single tree which covers the entire period of interest (including the end-of-month period). Both approaches will result in the same number of terminal nodes, but use of two trees may make it easier to keep track of information at each node coinciding with the end-of-month period (the terminal nodes in the primary tree). This information is used to determine the payoffs at all nodes that span the final 7 delivery days. Breaking the entire time period into two trees also simplifies the exposition of this procedure.
deliver bond at each node and the corresponding profits from substituting that bond in the delivery period.

The time-step in the secondary trees would ideally be chosen so that each coincides with the decision points at the end of all seven delivery days. Starting with the terminal nodes in the secondary tree, the incremental cashflow from delaying delivery can be computed by comparing the payoff at those nodes with the fixed payoff at the relevant terminal node in the primary tree.

More precisely, recall that the net cashflow received by the short position at the close of futures trading can be written as:

\[(F_T \cdot CF_i) - B_{i,T},\]  \hspace{1cm} (11)

where, \(F_T\) is the futures settlement price on the last trading day, \(CF_i\) is the conversion factor of the cheapest-to-deliver bond \(i\), and \(B_{i,T}\) is the cash bond price of the cheapest-to-deliver bond at time \(T\). The short will maximise the quantity in (11) by selecting the cheapest-to-deliver bond from the deliverable set and, because the unknown futures price is invariant for all bonds, the cheapest-to-deliver bond is dependent only on the various market prices and conversion factors. Rearranging (11), the cheapest-to-deliver bond at each terminal node on the primary tree can be identified as that which minimises:

\[B_{i,T}/CF_i.\]  \hspace{1cm} (12)

The cheapest-to-deliver bond at all nodes on the secondary tree can similarly be calculated and the payoff to the short position from delaying delivery for \(m\) days \((m = 1, \ldots, 7)\) computed as:

\[(B_{i,T}/CF_i) - (B_{i,T+m}/CF_i)\]  \hspace{1cm} (13)

Note that the quantity in (13) may be non-zero for two reasons. First and most obviously, innovations in the forward rate vector throughout the secondary tree will change the value of all delivery bonds at each node. Even if the same bond issue remains the cheapest-to-deliver, the payoffs to the short position will increase (decrease) if bond prices decrease (increase).
during the end-of-month period. Second, it is possible that the cheapest-to-deliver bond will change at some point during the last seven business days of the delivery month. The ability to substitute a cheaper bond in the settlement process confers a potentially valuable profit opportunity to the short position.

The values of the end-of-month options at all terminal nodes of the primary tree are estimated by first calculating the incremental profits from delaying delivery (using (13)), which are placed at the appropriate nodes in all of the secondary trees. Starting at the terminal nodes of a particular secondary tree, the value of the end-of-month option is set equal to the maturity payoffs. These are denoted as:

\[
V_{j, T}^i = \Delta B_{j, T}^i = (B_{T} / CF) - (B_{i, T+7} / CF),
\]  

where \(V_{j, T}^i\) is the node \(j\) value of the end-of-month option on the last day of the delivery period (day \(T + 7\)), and \(\Delta B_{j, T}^i\) is the difference between the price of the optimal bond at the close of trading in the futures contract and the price of the node \(j\) cheapest-to-deliver bond on day \(T + 7\). At all prior nodes, the short position has an early exercise decision based on a comparison of the incremental payoff from immediate delivery and the value of delaying delivery. That is, the value of the end-of-month option at all nodes between time \(T\) and \(T - 6\) is:

\[
V_{j, i}^i = \max[\Delta B_{j, i}, E(V_{T, i, i+1})], \quad i = 0, \ldots, 6.
\]  

where \(E(V_{T, i, i+1})\) is the discounted expected value of the delivery payoffs in the next period under the pseudo probabilities.

As \(V_{j, T}^i\) gives the total value of the end-of-month option (conditional on the starting forward rate vectors), the equilibrium futures prices at each terminal node in the primary tree is computed as the difference between the option-exclusive futures price and \(V_{j, T}^3\).

\[\text{It is important to note that this simple valuation approach will likely overstate the value of the end-of-month option. Recall that the futures price at the end of trading will be bid lower by the markets’ assessed value of the option. The approach outlined above initially assumes that the end-of-month option has no impact on the futures price, meaning that the computed profits to the short position during the final seven days will be overstated. The ‘true’ value of the option requires that the potential profits during the end-of-month period be based on ending futures prices which already reflect the value of the end-}\]
4.2 The Quality Option

Valuing the quality option is a straightforward process in the HL framework, and an appropriate method has already been alluded to in the discussion of the end-of-month option. Absent all other options, the equilibrium futures price should be directly related to the cheapest-to-deliver bond. At the maturity of the futures contract (at the end of trading in the contract, assuming no end-of-month option), the cheapest-to-deliver bond can be identified using (12): no-arbitrage requires that this also be the futures price. Thus, the futures price is set equal to the price of the cheapest-to-deliver bond at each terminal node, and the entire set of potentially optimal delivery bonds is then included in the current futures price via the recursive valuation procedure.

4.3 The Daily Wildcard Options

The daily options potentially create the most problems because each requires that an early exercise decision be evaluated at the end of each trading day in the delivery month. This in turn requires that the length of the time-step in the tree be chosen such that the end of each period coincides with the end of each day, and gives rise to an impossibly high number of total time-steps. Under the Ho and Lee parameterisation, imposing a one day time-step does not create an unworkably large number of terminal nodes.

The daily wildcard option can then be incorporated using a procedure similar to that of Fleming and Whaley (1994)\(^4\). That is, for all delivery days, construct a binomial tree such that a time-step coincides with both the end of futures trading (2 p.m. CST) and the end of the period by which the short must announce his intention to deliver (8 p.m. CST)\(^5\).

\(^4\) It is possible in this case to include the T-bond futures wildcard directly in the tree. Here, the wildcard period is 6 hours long (only 15 minutes for the OEX options), and thus only 4 time-steps are required for every day in the delivery period. The resulting total number of time-steps in a binomial model is not prohibitive, at least for the purposes of computing the wildcard value.

\(^5\) Previous studies differ in their choice of the length of the wildcard period. Some claim that bond trading ceases at 4 p.m. (CST), but Kane and Marcus (1986) point out that government bonds trade in a dealer market which is effectively open until the 8 p.m. delivery-notice deadline. Here, it is also assumed that the wildcard period is 6 hours long.
It is assumed here that in the absence of the end-of-month period, the wildcard option on the last day of trading has zero value - if the short has not closed out the position prior to the end of trading on this last day, then there is no option but to deliver. However, at all nodes corresponding to the delivery-notice deadline on the second to last trading day, the short has the option to either make delivery or to mark-to-market. Following Kane and Marcus (1986), that decision should be based on a comparison of the payoffs from the two actions:

\[ F_t - B_w \] \ldots profit from delivery

\[ F_t - F_{i+1} \] \ldots expected profit from marking-to-market

Here, \( F_t \) is the futures price at the end of trading (2 p.m.), \( F_{i+1} \) is the expected futures price at the close of trading on the next day (evaluated as the no-arbitrage futures price that would exist at the end of the wildcard period (8 p.m.) if the futures contract were trading), and \( B_w \) is the quoted price of the cheapest-to-deliver bond at the end of the wildcard period. If the node is reached as a result of a down (up) move in the Ho-Lee binomial framework, then both profits in (16) will be positive (negative) and the short will optimally choose the action which maximises the payoff (minimises the loss). When early delivery is indicated, the wildcard value can be computed as the difference between the two profits.

This apparently simple procedure is complicated by the difficulty in determining an appropriate futures price at the end of futures trading. Recall that when delivery is possible, the no-arbitrage futures price in the absence of any delivery options must be:

\[ F_t = B_w / CF_t \]  

(17)

where \( B_w \) is the quoted price of the cheapest-to-deliver bond and \( CF_t \) is the relevant conversion factor. However, at the beginning of the wildcard period on the second to last delivery day, the short has a wildcard opportunity whose value should be reflected in the futures price. That is,

\[ F_t = (B_w / CF_t) - WC_t \]  

(18)
where $WC_i$ is the value of the last wildcard option at the beginning of the wildcard period. Therein lies the problem: the objective is to determine the value of the wildcard option at all nodes corresponding to the end of the last wildcard period, but these values are themselves dependent on the unknown wildcard value at the beginning of the period.

The problem is dealt with as follows. At the end of the last wildcard period, the wildcard value at the corresponding nodes is computed using (16) and (17). The procedure then iterates back through the tree to the beginning of the previous wildcard period, initially setting the wildcard value at each node equal to the discounted expected value, based on the values at the appropriate up and down nodes. Then, at the end of the second to last wildcard period the wildcard value for that day is determined using (16) and (18). Here, the decision to mark-to-market or deliver is based on a comparison of payoffs which does reflect the fact that futures prices are bid lower by the value of the remaining wildcard options (other than the value of the wildcard option for that particular day).

If early exercise is indicated by the comparison of payoffs in (16), the wildcard value at each node is determined as the maximum of the profit from early exercise on that day, and the current value of the remaining wildcard opportunities. This reflects the fact that if the short chooses to deliver at the end of any particular wildcard period, he forgoes the value of all remaining wildcard opportunities. By working back through all earlier wildcard nodes, the procedure gives a cumulative wildcard value at the beginning of the delivery month.

Note that this procedure is likely to overvalue the wildcard option. While the futures prices at the beginning of each wildcard node are determined with reference to the wildcard values at all future nodes, they do not incorporate the values for the particular node being dealt with. This means that any profit (or loss minimisation) from an early exercise at the current node will be overstated.
5.0 Results

All of the simulations contained in this section are based on the following general set-up. We presume to be standing at the beginning of a delivery month, with 21 days until the end of trading in the futures contract and 28 days until the last possible delivery day. The option values are determined on an HL (binomial) tree that spans this period using the set of deliverable bonds (and their respective conversion factors) that were available for the March, 1996 contract. The only other required inputs are an initial term structure and an estimate of forward rate volatility. A range of initial forward curves was chosen with the intention of incorporating a number of different curve shapes; the analysis considers term structures that are flat, positive sloping, negative sloping, and humped. All are measured over the following intervals:

\[
\left[0, 0.083\right), \left(0.083, 0.25\right), \left(0.25, 0.5\right), \left(0.5, 0.75\right), \left(0.75, 1\right), (1, 2), (2, 3), (3, 5), (5, 7), (7, 10), (10, 14), (14, 20), (20, \infty) \right]
\]  

(19)

For each curve, option values are determined at three arbitrarily chosen volatility levels corresponding to approximately 10, 20 and 40 percent of the average forward rate. These levels are purposely chosen to be at the high end of what one may realistically observe in the market. Because option values are likely an increasing function of volatility, these choices should ensure that the computed values reflect the higher end of the possible range of ‘true’ values.

Initially, the values of the end-of-month, quality, and wildcard options are determined on an individual basis, exclusive of the impact of the other options. The cumulative value of all three options is then determined to provide some assessment of the relative importance of a particular option while controlling for the impact of the others.

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6 The deliverable set of bonds is actually the same for all of the contract months in 1996.
5.1 The End-of-Month Option

Estimates of the end-of-month delivery option were determined using the procedure outlined above, and are presented in Table 1. The values are computed as the difference between the option-exclusive and option-inclusive futures prices, as at the beginning of the hypothetical delivery month.

**Table 1**

**Simulated Values for the End of Month Delivery Option**

Each cell in the table gives the absolute value per $100 par (and value as a percentage of the computed, no-arbitrage futures price) of the option for a particular starting term structure of forward rates. The three values in each cell correspond to the low, medium, and high volatility cases.

<table>
<thead>
<tr>
<th>Initial Term Structure</th>
<th>Values 1</th>
<th>Values 2</th>
<th>Values 3</th>
<th>Values 4</th>
<th>Values 5</th>
<th>Values 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flat (^a)</td>
<td>0.097 (0.001)</td>
<td>0.107 (0.001)</td>
<td>0.089 (0.001)</td>
<td>0.148 (0.001)</td>
<td>0.130 (0.001)</td>
<td>0.124 (0.002)</td>
</tr>
<tr>
<td></td>
<td>0.130 (0.001)</td>
<td>0.120 (0.001)</td>
<td>0.120 (0.001)</td>
<td>0.111 (0.001)</td>
<td>0.111 (0.001)</td>
<td>0.107 (0.002)</td>
</tr>
<tr>
<td>Positive (^b)</td>
<td>0.055 (0.001)</td>
<td>0.079 (0.001)</td>
<td>0.091 (0.001)</td>
<td>0.099 (0.001)</td>
<td>0.103 (0.002)</td>
<td>0.109 (0.002)</td>
</tr>
<tr>
<td></td>
<td>0.070 (0.001)</td>
<td>0.079 (0.001)</td>
<td>0.091 (0.001)</td>
<td>0.099 (0.001)</td>
<td>0.103 (0.002)</td>
<td>0.109 (0.002)</td>
</tr>
<tr>
<td>Negative (^c)</td>
<td>0.232 (0.002)</td>
<td>0.232 (0.002)</td>
<td>0.219 (0.002)</td>
<td>0.200 (0.002)</td>
<td>0.211 (0.003)</td>
<td>0.197 (0.003)</td>
</tr>
<tr>
<td></td>
<td>0.232 (0.002)</td>
<td>0.232 (0.002)</td>
<td>0.219 (0.002)</td>
<td>0.200 (0.002)</td>
<td>0.211 (0.003)</td>
<td>0.197 (0.003)</td>
</tr>
<tr>
<td>Humped (^d)</td>
<td>0.086 (0.001)</td>
<td>0.107 (0.001)</td>
<td>0.089 (0.001)</td>
<td>0.104 (0.001)</td>
<td>0.129 (0.001)</td>
<td>0.136 (0.002)</td>
</tr>
<tr>
<td></td>
<td>0.079 (0.001)</td>
<td>0.101 (0.001)</td>
<td>0.089 (0.001)</td>
<td>0.088 (0.001)</td>
<td>0.129 (0.001)</td>
<td>0.136 (0.002)</td>
</tr>
<tr>
<td></td>
<td>0.061 (0.001)</td>
<td>0.083 (0.001)</td>
<td>0.089 (0.001)</td>
<td>0.054 (0.001)</td>
<td>0.128 (0.001)</td>
<td>0.135 (0.002)</td>
</tr>
</tbody>
</table>

\(^a\) The values in columns 1 through 6 correspond to initial term structures at 3, 5, 7, 8, 9, and 11 percent, respectively.

\(^b\) The six term structures with a positive slope begin at rates of 2.5, 4.5, 6.5, 7.5, 8.5, and 10.5 percent. Each assumes that the rates for the intervals specified in (19) increase by 0.5 of one percent.

\(^c\) The six term structures with a negative slope begin at rates of 8.5, 10.5, 12.5, 13.5, 14.5, and 16.5 percent. Each assumes that the rates for the intervals specified in (19) decrease by 0.5 of one percent.

\(^d\) The first two term structures are specified to have a pronounced positive hump at the long end of the curve. The second has the same shape as the first, but all rates are 2% higher. The third and fourth term structures both have a large positive hump at the short end of the term structure; again, the fourth curve is 2% higher than the third. The last two curves have a late negative hump and an early negative hump, respectively.

Previous research suggests that the main source of value for this option derives from the ability of the short position to optimally deliver a different bond during the end-of-month period compared to that which is indicated at the end of trading. The main conclusion that can be drawn from the tabulated results is that even when this factor is included, the value of the end-of-month option is uniformly small. This suggests that the assumed forward rate process
gives rise to little variability in the cheapest-to-deliver bond during the end-of-month period. Further, the estimates are insensitive to both the assumed forward rate volatility and the shape of the initial term structure.

Arak and Goodman (1987) estimate end-of-month option values on an ex-post basis for eight contract periods in the 1984-86 period. Although their value estimates are more variable across the considered contract periods, the results are generally similar in magnitude to those presented here.

5.2 The Quality Option

The value of the quality option is a function of uncertainty surrounding the eventual cheapest-to-deliver bond. Here, that value is estimated 90 days prior to the end of trading in the futures contract. It is computed as the difference between the futures price which assumes that the cheapest-to-deliver bond does not change over the estimation period and the futures price based on the appropriate cheapest-to-deliver bond at all of the terminal nodes in the tree.

Simulated values for the quality option are presented in Table 2. Consistent with the findings from most previous studies, the values are generally significant and commonly represent between 2 and 3 percent of the no-arbitrage futures price. As expected, the values are also increasing with the assumed volatility levels for all of the initial term structure shapes considered. One would anticipate that this results from the likelihood that a higher volatility induces a wider range of optimal delivery bonds at the end of the delivery period. However, some further analysis reveals that this is not the case; in almost all cases there are 3 or 4 bonds which are potentially the cheapest-to-deliver bond, and this number is not sensitive to either the shape of the initial term structure or the assumed volatility level. It appears that the value of the quality option is sensitive to both of these factors via their influence on the position in the tree at which the switch from one cheapest-to-deliver bond to another takes place. At low volatilities (and for particular initial term structures), the switch occurs at the extreme ends of the terminal branch in the tree and, as the probability associated with these changes is low, their impact on the value of the option is also low. Conversely, at higher volatility levels and for some other starting term structures, the switches occur at less extreme points in the terminal branch and thus have more impact on the computed value of the quality option.
Each cell in the table gives the absolute value per $100 par (and value as a percentage of the computed, no-arbitrage futures price) of the option for a particular starting term structure of forward rates. The three values in each cell correspond to the low, medium, and high volatility cases.

<table>
<thead>
<tr>
<th>Initial Term Structure</th>
<th>Values 1</th>
<th>Values 2</th>
<th>Values 3</th>
<th>Values 4</th>
<th>Values 5</th>
<th>Values 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flat *</td>
<td>1.140 (0.010)</td>
<td>1.386 (0.015)</td>
<td>0.978 (0.008)</td>
<td>1.500 (0.016)</td>
<td>0.582 (0.004)</td>
<td>1.046 (0.012)</td>
</tr>
<tr>
<td></td>
<td>1.138 (0.010)</td>
<td>1.382 (0.015)</td>
<td>0.984 (0.008)</td>
<td>1.497 (0.016)</td>
<td>0.611 (0.005)</td>
<td>1.047 (0.012)</td>
</tr>
<tr>
<td></td>
<td>1.129 (0.010)</td>
<td>1.367 (0.015)</td>
<td>1.083 (0.009)</td>
<td>1.482 (0.016)</td>
<td>0.883 (0.007)</td>
<td>1.126 (0.013)</td>
</tr>
<tr>
<td>Positive b</td>
<td>0.196 (0.000)</td>
<td>0.194 (0.000)</td>
<td>0.192 (0.000)</td>
<td>0.190 (0.000)</td>
<td>0.188 (0.000)</td>
<td>0.186 (0.000)</td>
</tr>
<tr>
<td></td>
<td>0.195 (0.000)</td>
<td>0.193 (0.000)</td>
<td>0.191 (0.000)</td>
<td>0.189 (0.000)</td>
<td>0.187 (0.000)</td>
<td>0.185 (0.000)</td>
</tr>
<tr>
<td></td>
<td>0.194 (0.000)</td>
<td>0.192 (0.000)</td>
<td>0.190 (0.000)</td>
<td>0.188 (0.000)</td>
<td>0.186 (0.000)</td>
<td>0.184 (0.000)</td>
</tr>
<tr>
<td>Negative c</td>
<td>0.000 (0.000)</td>
<td>0.947 (0.01)</td>
<td>0.677 (0.009)</td>
<td>0.639 (0.009)</td>
<td>0.638 (0.001)</td>
<td>0.695 (0.013)</td>
</tr>
<tr>
<td></td>
<td>0.796 (0.007)</td>
<td>0.979 (0.01)</td>
<td>0.680 (0.009)</td>
<td>0.641 (0.009)</td>
<td>0.639 (0.001)</td>
<td>0.696 (0.013)</td>
</tr>
<tr>
<td></td>
<td>0.000 (0.000)</td>
<td>0.971 (0.008)</td>
<td>2.081 (0.022)</td>
<td>2.108 (0.025)</td>
<td>1.456 (0.019)</td>
<td>1.044 (0.017)</td>
</tr>
<tr>
<td></td>
<td>0.018 (0.000)</td>
<td>0.984 (0.008)</td>
<td>2.265 (0.024)</td>
<td>2.394 (0.028)</td>
<td>1.579 (0.021)</td>
<td>1.047 (0.017)</td>
</tr>
<tr>
<td></td>
<td>0.156 (0.000)</td>
<td>1.169 (0.010)</td>
<td>2.829 (0.030)</td>
<td>2.758 (0.033)</td>
<td>2.030 (0.026)</td>
<td>1.175 (0.019)</td>
</tr>
<tr>
<td>Humped d</td>
<td>0.000 (0.000)</td>
<td>0.708 (0.004)</td>
<td>1.899 (0.017)</td>
<td>1.942 (0.018)</td>
<td>1.699 (0.015)</td>
<td>0.974 (0.013)</td>
</tr>
<tr>
<td></td>
<td>0.000 (0.000)</td>
<td>0.575 (0.004)</td>
<td>1.911 (0.017)</td>
<td>1.960 (0.018)</td>
<td>1.392 (0.015)</td>
<td>0.921 (0.013)</td>
</tr>
<tr>
<td></td>
<td>0.000 (0.000)</td>
<td>0.285 (0.004)</td>
<td>1.870 (0.017)</td>
<td>1.911 (0.017)</td>
<td>0.924 (0.011)</td>
<td>0.921 (0.013)</td>
</tr>
</tbody>
</table>

* The values in columns 1 through 6 correspond to initial term structures at 3, 5, 7, 8, 9, and 11 percent, respectively.

* The six term structures with a positive slope begin at rates of 2.5, 4.5, 6.5, 7.5, 8.5, and 10.5 percent. Each assumes that the rates for the intervals specified in (19) increase by 0.5 of one percent.

* The six term structures with a negative slope begin at rates of 8.5, 10.5, 12.5, 13.5, 14.5, and 16.5 percent. Each assumes that the rates for the intervals specified in (19) decrease by 0.5 of one percent.

* The first two term structures are specified to have a pronounced positive hump at the long end of the curve. The second has the same shape as the first, but all rates are 2% higher. The third and fourth term structures both have a large positive hump at the short end of the term structure; again, the fourth curve is 2% higher than the third. The last two curves have a late negative hump and an early negative hump, respectively.

It is also interesting to note that irrespective of the shape of the term structure, the option values peak when the initial term structure straddles the 8% yield level of the hypothetical bond that underlies the futures contract. This is consistent with previous research which points out that the uncertainty regarding the eventual cheapest-to-deliver bond is dependent on the level of the forward curve at delivery. It can be shown that at levels above and below 8%, the conversion factor system will tend to favour delivery of bonds with high and low duration. If the terminal forward curve is neither clearly above or below 8% then the characteristics of the likely optimal bond are more difficult to predict. In these circumstances, the increased option value implied by this valuation procedure again results from the position in the tree at which the change in the cheapest-to-deliver bond takes place.
5.3 The Wildcard Options

Estimates of the wildcard option (Table 3) are based on the procedure described in Section 4.3. They are generated as at the beginning of the delivery month, assuming that the end of trading in the futures contract occurs in 21 days. Importantly, it is initially assumed that the cheapest-to-deliver bond is identified at the beginning of the estimation period, and that that bond is then the only one available for delivery. By doing this, the value of the option is estimated without incorporating the potential advantage of optimally switching the delivery bond during the wildcard period. The indicated values are therefore simply a function of the size of the possible bond price changes during the wildcard period, which are themselves dependent on the assumed process for the forward curve.

Table 3
Simulated Values for the Wildcard Option

Each cell in the table gives the absolute value per $100 par (and value as a percentage of the computed, no-arbitrage futures price) of the option for a particular starting term structure of forward rates. The three values in each cell correspond to the low, medium, and high volatility cases.

<table>
<thead>
<tr>
<th>Initial Term Structure</th>
<th>Values 1</th>
<th>Values 2</th>
<th>Values 3</th>
<th>Values 4</th>
<th>Values 5</th>
<th>Values 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flat $^a$</td>
<td>0.022 (0.000)</td>
<td>0.026 (0.000)</td>
<td>0.027 (0.001)</td>
<td>0.035 (0.000)</td>
<td>0.032 (0.000)</td>
<td>0.028 (0.000)</td>
</tr>
<tr>
<td></td>
<td>0.045 (0.000)</td>
<td>0.056 (0.000)</td>
<td>0.058 (0.001)</td>
<td>0.066 (0.001)</td>
<td>0.062 (0.001)</td>
<td>0.053 (0.001)</td>
</tr>
<tr>
<td></td>
<td>0.092 (0.001)</td>
<td>0.113 (0.001)</td>
<td>0.118 (0.001)</td>
<td>0.129 (0.001)</td>
<td>0.119 (0.001)</td>
<td>0.101 (0.001)</td>
</tr>
<tr>
<td>Positive $^b$</td>
<td>0.029 (0.000)</td>
<td>0.027 (0.000)</td>
<td>0.024 (0.000)</td>
<td>0.023 (0.000)</td>
<td>0.022 (0.000)</td>
<td>0.020 (0.000)</td>
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<tr>
<td></td>
<td>0.055 (0.000)</td>
<td>0.051 (0.000)</td>
<td>0.046 (0.000)</td>
<td>0.044 (0.001)</td>
<td>0.041 (0.000)</td>
<td>0.037 (0.000)</td>
</tr>
<tr>
<td></td>
<td>0.107 (0.001)</td>
<td>0.099 (0.001)</td>
<td>0.089 (0.001)</td>
<td>0.084 (0.001)</td>
<td>0.079 (0.001)</td>
<td>0.070 (0.001)</td>
</tr>
<tr>
<td>Negative $^c$</td>
<td>0.032 (0.000)</td>
<td>0.032 (0.000)</td>
<td>0.030 (0.000)</td>
<td>0.031 (0.000)</td>
<td>0.034 (0.000)</td>
<td>0.029 (0.000)</td>
</tr>
<tr>
<td></td>
<td>0.068 (0.000)</td>
<td>0.068 (0.000)</td>
<td>0.064 (0.000)</td>
<td>0.059 (0.000)</td>
<td>0.065 (0.001)</td>
<td>0.054 (0.001)</td>
</tr>
<tr>
<td></td>
<td>0.137 (0.001)</td>
<td>0.138 (0.001)</td>
<td>0.129 (0.001)</td>
<td>0.115 (0.001)</td>
<td>0.013 (0.001)</td>
<td>0.103 (0.001)</td>
</tr>
<tr>
<td>Humped $^d$</td>
<td>0.023 (0.000)</td>
<td>0.029 (0.000)</td>
<td>0.027 (0.000)</td>
<td>0.027 (0.000)</td>
<td>0.029 (0.000)</td>
<td>0.029 (0.000)</td>
</tr>
<tr>
<td></td>
<td>0.048 (0.000)</td>
<td>0.056 (0.000)</td>
<td>0.057 (0.000)</td>
<td>0.056 (0.000)</td>
<td>0.061 (0.000)</td>
<td>0.055 (0.000)</td>
</tr>
<tr>
<td></td>
<td>0.099 (0.001)</td>
<td>0.109 (0.001)</td>
<td>0.116 (0.001)</td>
<td>0.115 (0.001)</td>
<td>0.123 (0.001)</td>
<td>0.106 (0.001)</td>
</tr>
</tbody>
</table>

$a$ The values in columns 1 through 6 correspond to initial term structures at 3, 5, 7, 8, 9, and 11 percent, respectively.

$b$ The six term structures with a positive slope begin at rates of 2.5, 4.5, 6.5, 7.5, 8.5, and 10.5 percent. Each assumes that the rates for the intervals specified in (19) increase by 0.5 of one percent.

$c$ The six term structures with a negative slope begin at rates of 8.5, 10.5, 12.5, 13.5, 14.5, and 16.5 percent. Each assumes that the rates for the intervals specified in (19) decrease by 0.5 of one percent.

$d$ The first two term structures are specified to have a pronounced positive hump at the long end of the curve. The second has the same shape as the first, but all rates are 2% higher. The third and fourth term structures both have a large positive hump at the short end of the term structure; again, the fourth curve is 2% higher than the third. The last two curves have a late negative hump and an early negative hump, respectively.
Under these conditions, the tabulated results indicate that the wildcard option is of little value to the party with the short position. While the values are universally increasing with the assumed level of volatility, they are generally insensitive to the shape of the starting forward curve and represent a maximum of only 0.10% of the option-exclusive futures price.

5.4 The Wildcard and Quality Options

The value of the wildcard options is re-estimated to account for the ability of the short to optimally deliver any of the deliverable set of bonds. This adjustment incorporates the possibility that the cheapest-to-deliver bond may change during the wildcard period. The potential for such a switch may be a significant source of value for the wildcard option. For example, assume that the level of the forward curve increases in the period between 2 p.m. and 8 p.m. to the extent that the optimal delivery bond at the end of the wildcard period is different from that indicated at the beginning of the period. If early delivery is optimal at the end of the wildcard period, the short receives a cashflow based on the ‘old’ delivery bond in return for delivery of the ‘new’, and cheaper, cheapest-to-deliver bond. The profit from early delivery in these circumstances can be substantial.

The new wildcard option values are presented in Table 4. When these are compared to those in Table 3 (where the cheapest-to-deliver bond is assumed to be fixed), it appears that the delivery flexibility adds little value for most of the starting forward curves considered. However, in those cases that the wildcard value is affected, the increase in value is often significant. Some examples include the fourth simulation for the flat, negative and humped term structure shapes where the wildcard value increases from a maximum of 0.1% of the futures prices to upwards of 7% of the futures price.
Each cell in the table gives the absolute value per $100 par (and value as a percentage of the computed, no-arbitrage futures price) of the option for a particular starting term structure of forward rates. The three values in each cell correspond to the low, medium, and high volatility cases.

### Table 4
Simulated Values for the Wildcard and Quality Options

<table>
<thead>
<tr>
<th>Initial Term Structure</th>
<th>Values 1</th>
<th>Values 2</th>
<th>Values 3</th>
<th>Values 4</th>
<th>Values 5</th>
<th>Values 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flat (a)</td>
<td>0.022 (0.000)</td>
<td>0.026 (0.000)</td>
<td>0.043 (0.001)</td>
<td>0.048 (0.000)</td>
<td>0.032 (0.000)</td>
<td>0.028 (0.000)</td>
</tr>
<tr>
<td></td>
<td>0.046 (0.000)</td>
<td>0.056 (0.000)</td>
<td>1.417 (0.013)</td>
<td>2.269 (0.023)</td>
<td>0.071 (0.001)</td>
<td>0.053 (0.001)</td>
</tr>
<tr>
<td></td>
<td>0.093 (0.001)</td>
<td>0.115 (0.001)</td>
<td>2.363 (0.021)</td>
<td>5.702 (0.058)</td>
<td>0.119 (0.001)</td>
<td>0.105 (0.001)</td>
</tr>
<tr>
<td>Positive (b)</td>
<td>0.029 (0.000)</td>
<td>0.027 (0.000)</td>
<td>0.024 (0.000)</td>
<td>0.023 (0.000)</td>
<td>0.022 (0.000)</td>
<td>0.020 (0.000)</td>
</tr>
<tr>
<td></td>
<td>0.830 (0.007)</td>
<td>0.051 (0.000)</td>
<td>0.046 (0.000)</td>
<td>0.044 (0.001)</td>
<td>0.041 (0.000)</td>
<td>0.037 (0.000)</td>
</tr>
<tr>
<td></td>
<td>3.727 (0.032)</td>
<td>0.101 (0.001)</td>
<td>0.089 (0.001)</td>
<td>0.084 (0.001)</td>
<td>0.079 (0.001)</td>
<td>0.070 (0.001)</td>
</tr>
<tr>
<td>Negative (c)</td>
<td>0.032 (0.000)</td>
<td>0.032 (0.000)</td>
<td>0.030 (0.000)</td>
<td>3.077 (0.036)</td>
<td>0.034 (0.000)</td>
<td>0.029 (0.000)</td>
</tr>
<tr>
<td></td>
<td>0.068 (0.000)</td>
<td>0.068 (0.000)</td>
<td>4.082 (0.043)</td>
<td>4.556 (0.054)</td>
<td>0.166 (0.001)</td>
<td>0.054 (0.001)</td>
</tr>
<tr>
<td></td>
<td>0.137 (0.001)</td>
<td>0.145 (0.001)</td>
<td>5.317 (0.056)</td>
<td>5.989 (0.071)</td>
<td>1.368 (0.018)</td>
<td>0.141 (0.002)</td>
</tr>
<tr>
<td>Humped (d)</td>
<td>1.917 (0.017)</td>
<td>0.029 (0.000)</td>
<td>0.027 (0.000)</td>
<td>0.027 (0.000)</td>
<td>0.029 (0.000)</td>
<td>0.029 (0.000)</td>
</tr>
<tr>
<td></td>
<td>3.112 (0.027)</td>
<td>0.056 (0.000)</td>
<td>0.057 (0.000)</td>
<td>2.730 (0.029)</td>
<td>0.061 (0.000)</td>
<td>0.055 (0.000)</td>
</tr>
<tr>
<td></td>
<td>0.332 (0.003)</td>
<td>1.183 (0.013)</td>
<td>0.122 (0.001)</td>
<td>5.851 (0.062)</td>
<td>0.162 (0.001)</td>
<td>0.133 (0.001)</td>
</tr>
</tbody>
</table>

\(a\) The values in columns 1 through 6 correspond to initial term structures at 3, 5, 7, 8, 9, and 11 percent, respectively.
\(b\) The six term structures with a positive slope begin at rates of 2.5, 4.5, 6.5, 7.5, 8.5, and 10.5 percent. Each assumes that the rates for the intervals specified in (19) increase by 0.5 of one percent.
\(c\) The six term structures with a negative slope begin at rates of 8.5, 10.5, 12.5, 13.5, 14.5, and 16.5 percent. Each assumes that the rates for the intervals specified in (19) decrease by 0.5 of one percent.
\(d\) The first two term structures are specified to have a pronounced positive hump at the long end of the curve. The second has the same shape as the first, but all rates are 2% higher. The third and fourth term structures both have a large positive hump at the short end of the term structure; again, the fourth curve is 2% higher than the third. The last two curves have a late negative hump and an early negative hump, respectively.

Similar to the explanation for the quality option, one can suggest that the wildcard option has value in these cases because a switch in the optimal delivery bond takes place toward the middle of the tree branches. The associated probability attached to this payoff is therefore higher than it would be if the switch in the cheapest-to-deliver bond occurs at more extreme points in the tree. The pattern of results is also largely consistent with that for the quality option. Estimated wildcard option values increase for starting term structures that are close to the 8% yield level because it is these conditions that induce a high probability of a change in the optimal delivery bond.

That the estimated wildcard values are significant in some circumstances can be viewed as problematic if the objective is to incorporate the impact of the option into a valuation procedure that uses a general HJM model. If the option accounts for an appreciable
proportion of the futures price, then any test of a pricing model that excludes the potential impact of the option will be biased. Unfortunately, as previously discussed, it is very difficult to efficiently incorporate the wildcard feature in the general HJM models.

One factor helps to alleviate these concerns. Recall that in the discussion of the valuation procedure used to estimate the wildcard delivery option, it was suggested that the given values are overstated. The option values are determined at the end of each wildcard period assuming that the futures price at the beginning of the period incorporates all of the subsequent wildcard opportunities, except for the value of the wildcard option at the node under consideration. Because the equilibrium futures price at the beginning of the wildcard period should be bid down by the value of all remaining wildcard opportunities, the profit to the short position computed using the procedure outlined here will be overstated. This will be especially evident at nodes for which a switch in the cheapest-to-deliver bond is likely.

For example, consider a node that corresponds to both the beginning of a wildcard period and a term structure for which a change in the optimal delivery bond is probable. As is the case in the valuation approach used here, assume for the moment that the futures price does not reflect this probability. If, at the subsequent up and down nodes, the term structure innovation is large enough to induce a change in the cheapest-to-deliver bond, then our pricing approach will likely indicate that exercise of the wildcard option is optimal. The short’s profit from doing so is computed as the difference between the fixed invoice price (based on the beginning of the period futures price) and the cost of purchasing the new cheapest-to-deliver bond at the end of the wildcard period. But, as the beginning futures price does not reflect the value of this wildcard opportunity (and is therefore higher than the ‘true’ price), estimates of wildcard values based on these profits will be exaggerated.
6.0 Summary and Concluding Remarks

For completeness, the aggregate value of all three delivery options is simultaneously estimated using a slightly modified version of the procedure used in the previous section. The modification includes the wildcard opportunity on the last day of trading, where the short’s decision is based on a comparison of the profit from early delivery to the value of the end-of-month option that would be foregone if delivery is made. As expected, the results are very similar to those presented in Table 5. However, although the aggregate option values are all higher than the combined value of the quality and wildcard options in each scenario, the difference is less than the estimated values of the relevant end-of-month options given in Table 1. This indicates that at the end of the trading period, the profit from exercising the last wildcard option exceeds the foregone value of the end-of-month option. At these nodes, the end-of-month option is therefore effectively worthless.

Overall, the option values presented above are consistent with those given in most previous studies. The simulations indicate that the quality option is the most valuable of the three delivery options considered here, and that this value is significant across all of the assumed starting term structures. Further, both the end-of-month and wildcard options have little value when estimated in isolation. However, when the wildcard opportunities are valued in the presence of the quality option, the wildcard values increase substantially for some particular interest rate scenarios. This result suggests that previous research may seriously understate the wildcard values because these studies typically assume that the optimal delivery bond is fixed at the start of the delivery month. Unfortunately, this finding also implies that an adequate pricing model for T-bond futures and futures options must consider the daily wildcard opportunities available to the short at the end of each delivery day.

Like all previous studies, this analysis has some important limitations. First, the estimated option values are determined using a specific valuation model that relies on a number of simplifying assumptions. Second, the specific valuation approaches used to value the quality, end-of-month, and wildcard options make some convenient assumptions that are likely to result in inflated option values. Third, the simulations presented in this section are far from exhaustive. They are based on a limited set of potential term structure shapes and levels, and only consider three arbitrarily chosen forward rate volatilities. The analysis is also based on
just one set of deliverable bonds (and conversion factors) - that which corresponds to the March 1996 delivery period.

Despite these problems, the evidence presented in this section generally supports the contention that an adequate pricing model for the T-bond futures contract can validly ignore the impacts of the end-of-month and wildcard options in many cases. Conversely, the quality option does seem to have a significant and consistent impact on futures prices and must therefore be included in any reasonable valuation model. Fortunately, incorporating these values into the lattices implied by any general HJM model is reasonably straightforward.

References


