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**COMPLEX DYNAMICS AND
ELEMENTARY ECONOMIC
THEORY**

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Abstract

Contrary to the view that complex dynamical systems are curiosities at a safe distance from core tenets of economics, it is shown that they undermine a few fundamental constructs and models. The long-run macroeconomic equilibrium defined as the limit of a sequence of short run equilibria is shown to be a problematic construct in simple aggregate models. The demonstration also undermines the very notion of the long run as used in mainstream economics, and the dictum that 'prices are flexible in the long run'.

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1. Introduction

Over the last decade and a half, a number of authors have explored the application of complex dynamic systems in economics. Initially the focus was on business cycles and finance literature, for example Benhabib and Day (1981), (1982), Boldrin (1988), Brock (1988), Grandmont (1985), Rosser, Jr.(1990). More recently a number of other areas, including the basic Walrasian exchange system have been subjected to analysis in works like Chiarella and Flaschel (1998); Dechert (1996); Majumdar and Mitra (1994),(1997); Montoro, Paz and Roig (1998), and Mukherji (1997). These works have been useful in illuminating some little-understood issues, and also in exposing the vulnerability of many traditional constructs.

The effect of this literature on the ‘modal’ economist is quite interesting. There is a general awareness that non-linear systems are curious, and they can explain many interesting natural phenomena. But in economics, they are thought to be somewhat contrived applications in specialised areas removed from the core tenets of the subject. The latter, comprising the basic models in micro and macro theory are considered robust in this respect. The purpose of this paper is to argue that even some of the simplest models of economic theory, the so-called robust ideas, are not as secure as they are thought to be. We analyse below a simplest one-sector macro model, used in 200-level textbooks as a basic description of the relation between the short run and the long run. We demonstrate that even with the well-behaved textbook assumptions, the relation between the short run and the long run contains problems serious enough to threaten this area of mainstream conceptualisation.

2. A Text-Book Model of Transition to the Long-Run

The following model is widely used in elementary textbooks to explain why short-run macro propositions are at variance with their long-run counterparts (see for example, Mankiw, 1997, Ch.12, Gordon, 1998, Ch.7). Short-run propositions are derived from a system with Keynesian features, capable of equilibrium unemployment and responsive to policy activism. Long run equilibrium on the other hand is classical, where equilibrium output (and employment) do not respond to policy. This contradiction is reconciled by imagining the long run as the limit of a sequence of appropriately defined short runs.

Short-run equilibria are generated from the interaction of the aggregate demand function and a short-run aggregate supply function. These equilibria may or may not coincide with the long-run equilibrium. In general they are expected to display Keynesian features. In that event, the short-run supply curve would not remain stable, but would start shifting over time. This generates a sequence of short-run equilibria expected to converge to the long-run equilibrium. Eventual convergence to the long run equilibrium implies that prices are flexible in the long run, even though they may be sticky in each single short-run period. The model also reconciles the possibility of short-run policy activism with its futility in the long run.

We will formalise this model, and then argue that (i) the long-run price dynamics converges to the long-run equilibrium only under very restrictive conditions. (ii) this restriction has no ready economic interpretation, and therefore it is impossible to judge if a real system is likely to satisfy it, and (iii) failing this restriction, the system shows complex dynamics and chaos.

The model consists of:

- (i) a long-run aggregate supply function (LAS), vertical at the ‘natural rate of output’ Y_n :

$$Y_l = Y_n \tag{1}$$

where Y_l is long-run supply and Y_n the ‘natural rate of output’.

- (ii) a short-run aggregate supply function (SAS), increasing in price level, P ,

$$Y_s = a + bP, b > 0 \tag{2}$$

where Y_s denotes short-run aggregate supply.

- (iii) an aggregate demand function (AD), falling in P . For simplicity it is taken to be unit elastic:

$$Y_d = \frac{\alpha}{P} \tag{3}$$

The long-run equilibrium price level for this model is $P^* = \frac{\alpha}{Y_n}$, given by equating (1) and

(3). Suppose the short-run equilibrium in some period t is not on the LAS, so that output and price level, respectively, $Y_{s,t}$ and P_t , differ from their long-run equilibrium values:

$$Y_{s,t} = a + bP_t = \frac{\alpha}{P_t} \neq Y_n, \text{ and } P_t \neq P^*.$$

These inequalities start off an inter-period price adjustment of the following form:

$$P_{t+1} = P_t + \lambda \left(\frac{\alpha}{P_t} - Y_n \right), \lambda > 0, \quad (4)$$

Following the usual tradition of price adjustment equations, (4) suggests that adjustment per period is proportional to the gap between current demand and Y_n .

Equations (1) through (4) formalise a wide variety of traditions in macroeconomics (see Mankiw, 1997, 333-342). The short-run supply equation (2) for example can be rationalised by sticky nominal wages (Gray, 1976; Fischer, 1977), sticky product prices (Ball, Mankiw and Romer, 1988), misperception of workers about their real wage (Friedman, 1968) or imperfect information (Lucas, 1977). Starting with different premises, all of them derive a a short-run supply function that fails to be vertical.

These traditions also posit an independent long-run vertical supply, and suggest a price adjustment like (4) as a transition mechanism from the short-run equilibrium to the long run.

Equation (4) will occupy our attention in the rest of the paper. We rewrite (4) in terms of an iterate function:

$$P_{t+1} = f(P_t), \text{ where } f(P) = P + \lambda \left(\frac{\alpha}{P} - Y_n \right), \quad (5)$$

The function $f(\cdot)$ has a minimum at $\bar{P} = \sqrt{\lambda\alpha}$. To ensure that all iterates of P_t are positive, we impose the restriction $f(\bar{P}) = 2\sqrt{\lambda\alpha} - \lambda Y_n > 0$, ie $4 > \frac{\lambda(Y_n)^2}{\alpha}$. The value of the

expression on the right hand side of this inequality plays a crucial role in the dynamics (4), and for easy reference we will denote it by c . Note that by definition $0 \leq c$.

Finally we get the equilibrium for the iterate function by putting $f(P) = P$ in (4), which gives $P = \alpha / Y_n$. We denote this by P^* .

3. Characterising the Dynamics

We will first characterise the dynamics of equation (4) for values of c in the permissible range, $0 \leq c \leq 4$.

Proposition 1: $c < 2 \Rightarrow P^*$ is locally stable for process (4); P^* is locally stable for process (4) $\Rightarrow c \leq 2$.

This proposition follows from the fact that at P^* the absolute value of the slope of the iterate function is less than unity (see Proposition 1, Saari, 1991).

$$f'(P^*) = 1 - \frac{\lambda\alpha}{P^{*2}} = 1 - c. \text{ Therefore, } c < 2 \text{ implies } |f'(P^*)| < 1$$

Proposition 2: For $2 < c < 2.5$, the path (4) is characterised by a stable 2-cycle.

2-cycles (stable or unstable) appear if there are P_1 and P_2 in the domain of P such that $f(P_1) = P_2$ and $f(P_2) = P_1$, implying that P iterates endlessly between P_1 and P_2 if it once attains either of these values. Such a pair P_1 and P_2 exist if

$$P_1 = P_2 + \lambda\left(\frac{\alpha}{P} - Y_n\right) \text{ and } P_2 = P_1 + \lambda\left(\frac{\alpha}{P_1} - Y_n\right) \quad (6)$$

Further, if P_1 and P_2 are attractors, then prices sooner or later are drawn to one of the values and the endless iteration ensues giving a stable 2-cycle. This situation arises if $|f'(P_1)f'(P_2)| < 1$.

$$\text{But, } |f'(P_1)f'(P_2)| = \left(1 - \frac{\lambda\alpha}{P_1^2}\right)\left(1 - \frac{\lambda\alpha}{P_2^2}\right) = 1 - \lambda\alpha\left(\frac{1}{P_1^2} + \frac{1}{P_2^2}\right) + \frac{\lambda^2\alpha^2}{P_1^2P_2^2}$$

Putting in values of $P_1 + P_2$ and P_1P_2 from above and simplifying, this gives

$$1 - 4\lambda\alpha\left(\frac{\lambda^2Y_n^2 - \lambda\alpha}{\lambda^2\alpha^2}\right) + 4\frac{\lambda^2\alpha^2}{\lambda^2\alpha^2} = 5 - 4\left(\frac{\lambda^2Y_n^2 - \lambda\alpha}{\lambda\alpha}\right) = 5 - 4\left(\frac{\lambda Y_n^2}{\alpha} - 1\right) = 9 - 4c$$

Since $|9 - 4c| < 1$ if and only if $2 < c < 2.5$, it follows that price and output will be attracted to a stable 2-cycle if $2 < c < 2.5$. Hence Proposition 2.

It should be clear from above that as c equals and then increases beyond 2.5, the qualitative nature of the dynamics should again change. In fact as c increases beyond 2.5, initially a stable 2²-cycle is born. Subsequently as it increases beyond another critical value, a stable 2³-cycle appears and so on. The distance between successive critical values of c falls rapidly.

Proposition 3: As λ increases in the range $2 < \lambda < 2.636$, the dynamics of (4) is characterised by stable 2ⁿ-cycles of higher and higher order.

For an illustrative demonstration, we will fix the value of all parameters except λ . We choose output units such that $Y_n = 1$ and assume $\alpha = 1$. Then $c = \lambda$. Since we have already seen (proposition 1) that P^* is locally stable for $c \leq 2$, and c has an upper bound at 4 for our problem, we will examine the range: $2 < \lambda < 4$.

Let λ_n denote the value of λ where a 2ⁿ-cycle appears for the first time. We have already seen that $\lambda_0 = 2$, and $\lambda_1 = 2.5$. It is a well-known result (see Lauwerier, 1986) that (1) the sequence λ_n converges to λ_∞ as n becomes large, and (2) for a one-dimensional iterate with a single parameter, there exists a constant F (the Feigenbaum constant = 4.669202...) which allows us to approximate the value of λ_∞ with the following relation:

$$\lambda_\infty \approx \frac{F\lambda_{n+1} - \lambda_n}{F - 1}$$

It follows from the above that $\lambda_\infty \approx 2.636269412259$

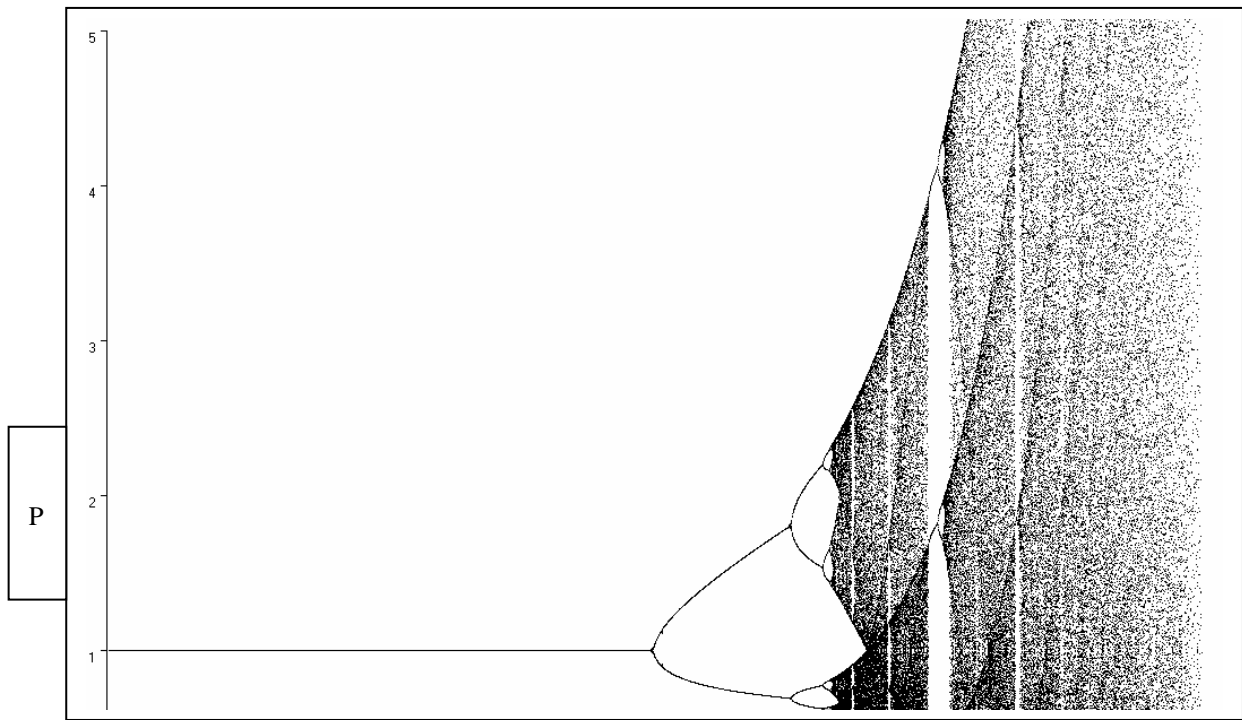
Thus as λ increases in the range $2 < \lambda < 2.636\dots$, the dynamics of (4) is characterised by stable 2^n -cycles of higher and higher order, implying that P and Y make endless iterations over a number of values that become infinitely large as λ approaches 2.636.

The interval beyond this value, ie $2.636\dots < \lambda < 4$ contains an infinity of values of λ for which the corresponding dynamics is characterised by a stable m -cycle. First such cycles to appear are of even periods. Next appear odd period stable cycles.(see Lauwerier,1986, p 45).

This range of stable cycles soon end, and beyond it there are no stable periodic orbits but there are an infinite number of unstable cycles. It can be shown that for values of λ past, but very close to, 3 the dynamics enters a chaotic phase.

Proposition 4: For $\lambda = 3.002$, the solution to (4) is chaotic.

We don't attempt to prove this proposition or discuss it heuristically. It can be established as an application of the theorem of Li-Yorke triple, due to Li and Yorke (1975). In stead we will illustrate the range of possible dynamics by referring to the bifurcation map for equation (4) in Figure 1.



Value of λ

Figure 1
Bifucation Map for Equation 4

On this diagram we have the values of λ on the x-axis. On the y-axis are plotted the iterates $f^n(P)$ for $n = 201$ to 300 , starting, for each λ , with a price chosen at random from the range:

$$[f(\bar{p}), \text{Max}\{\bar{p}, f(f(\bar{p}))\}]^1$$

The map shows stable equilibria for $\lambda \leq 2$, stable 2-cycles between $2 < \lambda \leq 2.5$. At 2.5, stable 4-cycles appear, and stable 8-cycles appear approximately around 2.6070 and so on. When λ reaches around 2.636..., we still have stable cycles but with an infinite number of values (2^∞ - cycles) shown by the grey patch starting around this value of λ .

Beyond this value, we encounter the diffused area of the map with small windows. The windows represent stable cycles of various order. About the diffused areas the possible conclusions are that either there are no stable cycles or there are stable cycles of such a long period that our iterations do not capture them.

¹ We do this because $f(p)$ for any p belonging to this closed interval also belongs to this interval.

4. Conclusions: Price Flexibility, Short-Runs and the Long-Run

Propositions of Section 3 produce serious difficulties not only for class-room discourses but also for mainstream macro theory.

First of all convergence to long-run classical equilibrium (P^*, Y_n) is conditional. This challenges the general belief that equilibrium at less than the natural rate is only a short-run phenomenon. It also undermines the idea of long-run price flexibility, which is a major assumption in all models of growth and long-run behaviour. At the same time it opens up the possibility that empirical time series data may not always come from an exponential growth-plus-shocks type of data generation process. The data generation process may be far more complex, and current econometric practices may be widely misleading.

Secondly, the condition for convergence depends not only on the (family of) supply functions but also on the demand function. P changes from one period to the next because of a shift in the short-run supply function. This shift produces a price change, whose extent also depends on the parameters of the AD function. λ captures the overall effect on P of an inter-period shift of the short-run supply function, and thus incorporates features of the supply as well as the demand side. For example, it can be checked that if the aggregate demand elasticity is e , ie equation (3) is replaced with $Yd = \frac{\alpha}{P^e}$, then the restriction for convergence is $0 < \lambda < 2/e$.

This runs counter to the intuitive notion of price flexibility, which is considered a feature of the supply side. Because λ does not have an interpretation involving either demand or supply alone, it is difficult to weave a simple economic story around the condition of convergence and long-run price flexibility.

Thirdly, if we propose λ as a measure of price flexibility we run counter to an established usage. In this case, flexibility greater than a critical value implies that the system would *not* converge to the long-run equilibrium. This goes against the normal use of phrases like flexibility and rigidity in economics, which would have us expect that the stickier the prices, the more difficult it is to attain the long-run equilibrium or the natural rate.

We may conclude by noting that the notion of the long run as a limit of a sequence relies on the convergence of the defined sequence. While this convergence is taken for granted, it is actually conditional on parameters not understood very well. At least two sets of so-called robust constructs are in threat here. The first is the intuition that prices are flexible in the long run, and its corollary that even though we observe Keynesian outcomes in the short run, the system is inexorably travelling towards a classical equilibrium. The second is the very construct of the long run as it is used in mainstream economics. If it can not be defined before specifying a model, obviously it creates a serious methodological problem.

References

- Ball, L., Mankiw, G., and Romer, D. (1988) The New Keynesian Economics and the Output Inflation Tradeoff, *Brookings Papers on Economic Activity* (1988:1), 1-65.
- Dechert, W. Davis (1996) ed. *Chaos theory in economics: Methods, models and evidence*, Elgar Reference Collection, International Library of Critical Writings in Economics, no. 66. Cheltenham, U.K, Elgar.
- Fischer, S. (1977) Long-term Contracts, Rational Expectations, and the Optimal Money Supply Rule, *Journal of Political economy*, 85, 191-205.
- Friedman, M. (1968) The Role of Monetary Policy, *American Economic Review*, 58, 1-17.
- Gordon, R. J. (1998) *Macroeconomics*, Seventh Edition, Addison-Wesley Longman Inc., Reading, MA.
- Gray, A. J. (1976) Wage Indexation: A Macroeconomic Approach, *Journal of Monetary Economics*, 2, 221-235.
- Lauwerier, H.A. (1986) One Dimensional Iterative Maps, in *Chaos*, (Ed) A.V. Holden, Manchester University Press, 39-57.
- Li, T., and Yorke, J.A. (1975) Period Three implies Chaos, *American Mathematical Monthly*, 82, 985-992.
- Lucas, R. E, Jr. (1977) Understanding Business Cycles, in *Stabilisation of the Domestic and the International Economy*, vol 5 of Carnegie-Rochester Conference on Public Policy, Amsterdam, North Holland Publishing Company.
- Majumdar, M and Mitra, T. (1994) Periodic and Chaotic Programs of Optimal Intertemporal Allocation in an Aggregative Model with Wealth Effects, *Economic Theory*; V4 N5, 649-76.
- Majumdar, M and Mitra, T. (1997) Complexities of Concrete Walrasian Systems in *Issues in Economic Theory and Policy, Essays in Honour of Professor Tapas Majumdar*, (Eds) A. Bose, et al, Oxford University Press, Delhi, 41-63.
- Mankiw, G. N. (1997) *Macroeconomics*, Third Edition, Worth Publishers, New York.
- Mukherji, A. (1997) Discrete Processes, DSA Working paper, *Centre for Economic Studies and Planning*, Jawaharlal Nehru University.
- Saari, D.G. (1991) Erratic Behaviour in Economic Models, *Journal of Economic Behaviour and Organisation*, 16, 3-36.